Comment on "Ward identities for transport of classical waves in disordered media"

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Recently Nieh *et al.* [Phys. Rev. E **57**, 1145 (1998)] have considered a version of derivation of the Ward identities for scalar and vector classical wave field propagation in random media and noted that their results are in contradiction with those obtained by Barabanenkov and Ozrin [Phys. Lett. A **154**, 38 (1991)]. In this Comment we show that the derivation given by Nieh *et al.* is based on an incorrect equation for the energy-vertex function where the term that takes into account the contributions of the scatterer polarization to the field energy is lost. Restoring this term removes the above-noted contradiction.

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Ward identities (WI) for classical and quantum fields in

random media relate the irreducible self-energy (two-point) functions to the irreducible vertex (four-point) functions and, in these terms, represent the conservation laws. In the case of electron-impurity interaction there is a perturbative, diagram approach to derivation of the WI [1], which utilizes the idea that the irreducible vertex has a topological structure of the functional derivative of the mass operator with respect to the exact Green function. In the current decade generalized WI for multiple scattering of classical scalar [2,3] and vector electromagnetic [4,5] waves in random media have been derived with the aid of an algebraic method. The principal difference between the WI for classical waves and that for the electron transport is connected to that the effective potential for wave scattering is frequency dependent. Recently another version of the WI for classical wave multiple scattering in random media was considered based on a modified Taka-

In the present Comment we show that this version of the WI is erroneous because it was obtained in [6] from an incorrect equation for the "energy-vertex" function. We present the correct equation for this function, which leads to the WI found earlier in Refs. [2,3]. For precision, we consider in detail the case of scalar waves.

hashi approach [6] (see also [7]). This version of the WI

seems strange being inconsistent with the perturbation ex-

pansion in power series in the effective scattering potential.

The time Fourier transform of a scalar field $\varphi({\bf r},\Omega)$ satisfies the Helmholtz equation

$$[\epsilon(\mathbf{r})\Omega^2 + \Delta]\varphi(\mathbf{r},\Omega) = Q(\mathbf{r},\Omega), \qquad (1)$$

where $Q(\mathbf{r},\Omega)$ is the Fourier transform of a source, the dielectric constant is $\epsilon(\mathbf{r}) = 1 + \delta \epsilon(\mathbf{r})$ with the random part $\delta \epsilon$ that is formed, for instance, by a system of identical dielectric scatterer spheres, randomly distributed in the space. The field, generated by the source Q, can be represented in the form

$$\varphi(\mathbf{r},\Omega) = \int d\mathbf{r}' \mathcal{G}(\mathbf{r},\mathbf{r}';\Omega+i0) Q(\mathbf{r}',\Omega), \qquad (2)$$

where the Green function $\mathcal{G}(\mathbf{r},\mathbf{r}';\omega)$ obeys equation

$$[\omega^2 + \Delta - V(\mathbf{r}, \omega)] \mathcal{G}(\mathbf{r}, \mathbf{r}'; \omega) = \delta(\mathbf{r} - \mathbf{r}'). \tag{3}$$

Here ω is, in general, the complex frequency with a value depending on choosing the Green function to be the retarded, advanced, or causal one; $V(\mathbf{r}, \omega)$ is given by

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$$V(\mathbf{r}, \omega) = -g(\omega) \delta \epsilon(\mathbf{r}), \quad g(\omega) = \omega^2$$
 (4)

and referred to as a scattering or random "potential." A specific designation for the frequency-dependent factor $g(\omega)$ in the "potential" is to compare the final results for scalar wave fields with those for the case of the quantum-mechanical electron-impurity system: a formal analogy between these physical models is seen through the replacement $g(\omega) \rightarrow 1, \omega^2 \rightarrow E$ in Eqs. (3) and (4).

We will deal with the ensemble-averaged two-point and four-point Green functions given by $G(1;\omega_1) = \langle \mathcal{G}(\mathbf{r}_1,\mathbf{r}_1';\omega_1) \rangle$ and $F(1,2;\omega_1,\omega_2) = \langle \mathcal{G}(\mathbf{r}_1,\mathbf{r}_1';\omega_1)\mathcal{G}(\mathbf{r}_2,\mathbf{r}_2';\omega_2) \rangle$. Since averaging restores the translational invariance, these functions depend on differences between coordinates, and their space Fourier transforms can be defined by

$$G_p(\omega) = \int d\mathbf{r} \exp[-i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')]\langle \mathcal{G}(\mathbf{r},\mathbf{r}';\omega)\rangle,$$

$$F_{pp'}(\mathbf{q}; \boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{2}) = \int d\mathbf{R} d\mathbf{r} d\mathbf{r}' \exp[-i\mathbf{q} \cdot (\mathbf{R} - \mathbf{R}') - i\mathbf{p} \cdot \mathbf{r} + i\mathbf{p}' \cdot \mathbf{r}'] \langle \mathcal{G}(\mathbf{R} + \mathbf{r}/2, \mathbf{R}' + \mathbf{r}'/2; \boldsymbol{\omega}_{1}) \times \mathcal{G}(\mathbf{R} - \mathbf{r}/2, \mathbf{R}' - \mathbf{r}'/2; \boldsymbol{\omega}_{2}) \rangle.$$
(5)

Here $G_p(\omega)$ and $F_{pp'}(\mathbf{q};\omega_1,\omega_2)$ satisfy, respectively, the Dyson and Bethe-Salpeter (BSE) equations represented in the form

$$G_{p}(\omega) = [\omega^{2} - p^{2} - M_{p}(\omega)]^{-1},$$
 (6)

$$F_{p'p}(\mathbf{q}; \omega_1, \omega_2) = F_p^{(0)}(\mathbf{q}; \omega_1, \omega_2)$$

$$\times \left(\delta_{p'p} + \int_{p''} F_{p'p''}(\mathbf{q}; \omega_1, \omega_2) \right)$$

$$\times K_{p''p}(\mathbf{q}; \omega_1, \omega_2) , \qquad (7)$$

where $F_p^{(0)}(\mathbf{q};\omega_1,\omega_2) = G_{p_+}(\omega_1)G_{p_-}(\omega_2)$, $\mathbf{p}_{\pm} = \mathbf{p} \pm \mathbf{q}/2$. In Eq. (7), the convention is adopted that $\int_p = (2\pi)^{-3} \int d\mathbf{p}$, and $\delta_{pp'} = (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}')$. The functions $M_p(\omega)$ and $K_{pp'}(\mathbf{q},\omega_1,\omega_2)$ are usually referred to as the mass operator (or irreducible self-energy function) and the irreducible (four-point) vertex function, respectively. Due to the reciprocity condition, the four-point Green function and the corresponding irreducible vertex satisfy the symmetry relations given by $F_{pp'}(\mathbf{q};\omega_1,\omega_2) = F_{p'p}(\mathbf{q};\omega_1,\omega_2)$ and $K_{pp'}(\mathbf{q};\omega_1,\omega_2)$.

In the case of classical waves, a simple approach to derivation of WI can be based on a close relationship between the Green functions, G and F, that does not include the "potential" in the explicit form.

Let us consider the function

$$\Psi(1,2;\omega_1,\omega_2) = \langle V(\mathbf{r}_1,\omega_1)\mathcal{G}(\mathbf{r}_1,\mathbf{r}_1';\omega_1)\mathcal{G}(\mathbf{r}_2,\mathbf{r}_2';\omega_2) \rangle. \tag{8}$$

From the definition of the random potential given by Eq. (4), a "scaling" property is easily seen as

$$g(\omega_1)V(\mathbf{r},\omega_2) = g(\omega_2)V(\mathbf{r},\omega_1).$$
 (9)

Therefore, the function Ψ satisfies the identity

$$\lim_{\mathbf{r}_{1} \to \mathbf{r}_{2}} [g(\omega_{2})\Psi(1,2;\omega_{1},\omega_{2}) - g(\omega_{1})\Psi(2,1;\omega_{2},\omega_{1})] = 0.$$
(10)

On the other hand, according to Eq. (3), Ψ obeys the equation

$$\Psi(1,2;\boldsymbol{\omega}_1,\boldsymbol{\omega}_2) = (\boldsymbol{\omega}_1^2 + \boldsymbol{\Delta}_1)F(1,2;\boldsymbol{\omega}_1,\boldsymbol{\omega}_2) - \delta(\mathbf{r}_1 - \mathbf{r}_1')G(2;\boldsymbol{\omega}_2). \tag{11}$$

Substituting Eq. (11) into the left-hand side (lhs) of identity (10) results in a desirable relationship between F and G with the Fourier transform given by

$$\int_{p'} [g(\omega_2)(\omega_1^2 - p'_+^2) - g(\omega_1)(\omega_2^2 - p'_-^2)] F_{p'p}(\mathbf{q}; \omega_1, \omega_2)
= g(\omega_2) G_{p_-}(\omega_2) - g(\omega_1) G_{p_+}(\omega_1),$$
(12)

where $\mathbf{p}_{+} = \mathbf{p} \pm \mathbf{q}/2$ and $\mathbf{p}'_{+} = \mathbf{p}' \pm \mathbf{q}/2$.

In order to transform Eq. (12) to a relation between the irreducible functions, we introduce the (three-point) vertex function γ defined by

$$\gamma_{p}(\mathbf{q};\omega_{1},\omega_{2}) = \int_{p'} \gamma_{p'}^{(0)}(\mathbf{q};\omega_{1},\omega_{2}) F_{p'p}(\mathbf{q};\omega_{1},\omega_{2})$$
$$\times [F_{p}^{(0)}(\mathbf{q};\omega_{1},\omega_{2})]^{-1}$$
(13)

with

$$\gamma_p^{(0)}(\mathbf{q};\omega_1,\omega_2) = g(\omega_2)G_{\mathbf{p}_+}^{(0)-1}(\omega_1) - g(\omega_1)G_{\mathbf{p}_-}^{(0)-1}(\omega_2), \tag{14}$$

where $G_{\bf p}^{(0)}(\omega) = (\omega^2 - p^2)^{-1}$. According to Eq. (12), γ satisfies the identity

$$\gamma_{p}(\mathbf{q};\omega_{1},\omega_{2}) = g(\omega_{2})G_{p_{+}}^{-1}(\omega_{1}) - g(\omega_{1})G_{p_{-}}^{-1}(\omega_{2}).$$
(15)

On the other hand, substituting the BSE given by Eq. (7) into the right-hand side (rhs) of Eq. (13) yields the equation

$$\gamma_{p}(\mathbf{q};\omega_{1},\omega_{2}) = \gamma_{p}^{(0)}(\mathbf{q};\omega_{1},\omega_{2}) + \int_{p'} \gamma_{p'}(\mathbf{q};\omega_{1},\omega_{2})$$
$$\times F_{p'}^{(0)}(\mathbf{q};\omega_{1},\omega_{2})K_{p'p}(\mathbf{q};\omega_{1},\omega_{2}). \quad (16)$$

Substituting the identity (15) into the both sides of this equation with use of the Dyson equation (6) results in the following final relationship [2]:

$$g(\omega_{2})M_{p_{+}}(\omega_{1}) - g(\omega_{1})M_{p_{-}}(\omega_{2})$$

$$= \int_{p'} K_{pp'}(\mathbf{q}; \omega_{1}, \omega_{2})$$

$$\times [g(\omega_{1})G_{p'}(\omega_{1}) - g(\omega_{2})G_{p'}(\omega_{2})]. \quad (17)$$

The relation (17), derived, as is seen, without any approximations can be regarded as a scalar field version of the WI, which is true for arbitrary momenta $\mathbf{p}_+, \mathbf{p}_-$ and complex frequencies ω_1, ω_2 . Note that the formal replacement $g(\omega) \to 1, \omega^2 \to E$ reduces Eq. (12) to the well-known version of the continuity equation, and Eq. (17) to the conventional WI used in the case of the electron-impurity interaction [1].

In order to analyze the approach developed by Nieh *et al.* [6] we need to formulate a scalar field version of the Poynting theorem written in terms of the Green function.

In the case of classical scalar field, the spectral densities of the energy density and the energy flux density (Poynting vector) averaged over ensemble are defined, respectively, by [8]

$$W = \frac{1}{2} \langle \Omega_{+} \Omega_{-} \boldsymbol{\epsilon} \varphi^{*} \varphi + \boldsymbol{\nabla} \varphi^{*} \cdot \boldsymbol{\nabla} \varphi \rangle, \tag{18}$$

$$\mathbf{S} = \frac{i}{2} \langle \Omega_{+} \varphi \nabla \varphi^{*} - \Omega_{-} \varphi^{*} \nabla \varphi \rangle, \tag{19}$$

where the complex frequencies are given by $\Omega_{\pm} = \Omega \pm (\omega + i0)/2$, the time Fourier transforms of the fields are denoted as $\varphi \equiv \varphi(\mathbf{r}, \Omega_{+})$ and $\varphi^* \equiv \varphi^*(\mathbf{r}, \Omega_{-})$, and the same nota-

tions are used for the source $Q(\mathbf{r},\Omega)$ related to field by Eq. (2). Thus, according to Eq. (1), the energy conservation law can be written as

$$i\omega W - \operatorname{div} \mathbf{S} = \frac{i}{2} \langle \Omega_{-} \varphi^* Q - \Omega_{+} Q^* \varphi \rangle,$$
 (20)

where $\omega = \Omega_{+} - \Omega_{-}$.

There are two ways to derive the equation that represents the conservation law in terms of the Green functions. The first, straightforward way is based on Eq. (2) and consists of calculating the second functional derivative of the lhs and the rhs of Eq. (20) with respect to the source O.

The second (and maybe, more simple) way utilizes an observation that Eq. (20) can be obtained by averaging of the part-by-part difference between Eq. (1) for $\varphi(\mathbf{r},\Omega_+)$ multiplied by the factor $\Omega_-\varphi^*(\mathbf{r},\Omega_-)$ and Eq. (1) for $\varphi^*(\mathbf{r},\Omega_-)$ multiplied by the factor $\Omega_+\varphi(\mathbf{r},\Omega_+)$. In terms of the Green functions, such a procedure means calculating the part-by-part difference between Eq. (3) for $\mathcal{G}(\mathbf{r}_1,\mathbf{r}_1';\omega_1)$ multiplied by the factor $\omega_2\mathcal{G}(\mathbf{r}_2,\mathbf{r}_2';\omega_2)$ and Eq. (3) for $\mathcal{G}(\mathbf{r}_2,\mathbf{r}_2';\omega_2)$ multiplied by the factor $\omega_1\mathcal{G}(\mathbf{r}_1,\mathbf{r}_1';\omega_1)$. Hence, the Greenfunction equivalent of the conservation law takes the form

$$\lim_{\mathbf{r}_{1}\to\mathbf{r}_{2}} \{ [\omega_{2}(\Delta_{1}+\omega_{1}^{2})-\omega_{1}(\Delta_{2}+\omega_{2}^{2})]F(1,2;\omega_{1},\omega_{2}) - \omega_{12}[\Psi(1,2;\omega_{1},\omega_{2})-\Psi(2,1;\omega_{2},\omega_{1})] \}$$

$$= \lim_{\mathbf{r}_{1}\to\mathbf{r}_{2}} [\omega_{2}\delta(\mathbf{r}_{1}-\mathbf{r}_{1}^{\prime})G(2;\omega_{2})-\omega_{1}\delta(\mathbf{r}_{2}-\mathbf{r}_{2}^{\prime})G(1;\omega_{1})],$$
(21)

where $\omega_{12} = \omega_1 \omega_2 / (\omega_1 + \omega_2)$, ω_1 and ω_2 are the complex frequency, in particular, $\omega_{1,2} = \Omega_{\pm}$, the function Ψ is defined by Eq. (8). Note that here we make use of the equality

$$\lim_{\mathbf{r}_{1} \to \mathbf{r}_{2}} \left[\omega_{2} \Psi(1,2;\omega_{1},\omega_{2}) - \omega_{1} \Psi(2,1;\omega_{2},\omega_{1}) \right]
= \omega_{12} \lim_{\mathbf{r}_{1} \to \mathbf{r}_{2}} \left[\Psi(1,2;\omega_{1},\omega_{2}) - \Psi(2,1;\omega_{2},\omega_{1}) \right],$$
(22)

which follows directly from the precursor to the WI Eq. (10). It is seen that the first pair of terms in the lhs of Eq. (21) including the function F corresponds term by term to the lhs of the energy conservation law given by Eqs. (18)–(20) with the exception of the term in the energy density (18) proportional to the random part of the dielectric constant, $\delta\epsilon$, related to the scatterer polarization. This part of the dielectric constant is taken into account in the second pair of terms in the lhs of Eq. (21) including the function Ψ .

The Green-function representation of the conservation law given by Eq. (21) can be used as a starting point for alternative derivation of the WI. Indeed, the Fourier transform of this equation multiplied by the factor $[G(1;\omega_1)G(2;\omega_2)]^{-1}$ can be represented as

$$\tilde{\gamma}_p(\mathbf{q};\omega_1,\omega_2) = \omega_2 G_{p_+}^{-1}(\omega_1) - \omega_1 G_{p_-}^{-1}(\omega_2),$$
 (23)

where $\mathbf{p}_{+} = \mathbf{p} \pm \mathbf{q}/2$ and

$$\widetilde{\gamma}_{p}(\mathbf{q};\omega_{1},\omega_{2}) = \gamma_{p}^{(e)}(\mathbf{q};\omega_{1},\omega_{2}) + \gamma_{p}^{(v)}(\mathbf{q};\omega_{1},\omega_{2}). \quad (24)$$

The functions $\gamma^{(e)}$ and $\gamma^{(v)}$, corresponding, respectively, to the first and the second pairs of terms in the lhs of Eq. (21), are defined by

$$\gamma_p^{(e,v)}(\mathbf{q};\omega_1,\omega_2) = \int_{p'} \gamma_{p'}^{(e,v0)}(\mathbf{q};\omega_1,\omega_2) F_{p'p}(\mathbf{q};\omega_1,\omega_2) \\
\times [F_p^{(0)}(\mathbf{q};\omega_1,\omega_2)]^{-1},$$
(25)

where

$$\gamma_p^{(e0)}(\mathbf{q};\omega_1,\omega_2) = \omega_2 G_{\mathbf{p}_+}^{(0)-1}(\omega_1) - \omega_1 G_{\mathbf{p}_-}^{(0)-1}(\omega_2). \tag{26}$$

The explicit expression for $\gamma^{(v0)}$ found from Eqs. (6), (7), and (11) is of the form

$$\gamma_{p}^{(v0)}(\mathbf{q};\omega_{1},\omega_{2}) = 2\omega_{12} \left(\Delta M_{p}(\mathbf{q};\omega_{1},\omega_{2}) - \int_{p'} \Delta G_{p'}(\mathbf{q};\omega_{1},\omega_{2}) K_{p'p}(\mathbf{q};\omega_{1},\omega_{2}) \right)$$

$$(27)$$

with $\Delta G_p(\mathbf{q}; \omega_1, \omega_2) = \frac{1}{2} [G_{\mathbf{p}-\mathbf{q}/2}(\omega_2) - G_{\mathbf{p}+\mathbf{q}/2}(\omega_1)]$, and $\Delta M_p(\mathbf{q}; \omega_1, \omega_2)$ being defined in the same manner.

Substituting the BSE given by Eq. (7) into the rhs of Eqs. (25) allows us to obtain the integral equation of the type (16),

$$\widetilde{\gamma}_{p}(\mathbf{q},\omega_{1},\omega_{2}) = \widetilde{\gamma}_{p}^{(0)}(\mathbf{q},\omega_{1},\omega_{2}) + \int_{p'} \widetilde{\gamma}_{p'}(\mathbf{q},\omega_{1},\omega_{2})$$

$$\times F_{p'}^{(0)}(\mathbf{q};\omega_{1},\omega_{2})K_{p'p}(\mathbf{q},\omega_{1},\omega_{2}) \quad (28)$$

with

$$\widetilde{\gamma}_p^{(0)}(\mathbf{q}, \omega_1, \omega_2) = \gamma_p^{(e0)}(\mathbf{q}, \omega_1, \omega_2) + \gamma_p^{(v0)}(\mathbf{q}; \omega_1, \omega_2). \tag{29}$$

Finally, it can be shown that substitution of the equality (23) into the both sides of Eq. (28) leads, after simple algebra, to the WI, Eq. (17). It is worth emphasizing that the key point of this WI derivation is the use of the equality (22), which follows from the "scaling" property (9) of the scattering "potential."

Now we can compare the above equations with those used by Nieh *et al.* [6]. They declare the quantum field theory to be the basis for the approach and start with calculating the three-point four-vector function

$$N_{\mu}(x;y,z) \equiv \langle 0 | T(T_{\mu 0}(x)\varphi(y)\varphi(z)) | 0 \rangle \tag{30}$$

and the energy-vertex function $\Gamma_{\mu}(y|x|z)$ defined by

$$\langle N_{\mu}(x;y,z)\rangle = -\int dy'dz'G(y-y')G(z-z')\Gamma_{\mu}(y'|x|z'). \tag{31}$$

Here $|0\rangle$ represents the ground state of the scalar field, T denotes time ordering, $T_{\mu 0}(x)$ is the energy-momentum tensor; the four-dimensional notation $x^{\mu} = (t, \mathbf{r})$ is used with the greek subscript (superscript) denoting four-vector index running over 0,1,2,3.

Comparing Eq. (23) with Eq. (28) from Ref. [6] one can conclude that the function $\tilde{\gamma}_p(\mathbf{q},\omega_1,\omega_2)$ coincides with the time-space Fourier transform of the type (5) of the four-divergence of the energy vertex functions, $-i\partial_x^\mu\Gamma_\mu(y|x|s)$. Moreover, straightforward calculations based on the Wick theorem show that the time Fourier transform of the four-divergence $i\partial_x^\mu\langle N_\mu\rangle$ coincides with the lhs of Eq. (21). Of

course, in accordance with the quantum field theory rules (see, e.g., Ref. [9]) the energy-momentum tensor in Eq. (30) should be taken in the form of normal product $:T_{u0}(x):$.

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However, according to Nieh *et al.* [6], the time-space Fourier transform of the four-divergence $-i\partial_x^\mu\Gamma_\mu(y|x|s)$ satisfies the equation [see Eq. (30) in [6]] of the type (28) with the function $\gamma_p^{(e0)}(\mathbf{q},\omega_1,\omega_2)$ erroneously replacing $\widetilde{\gamma}_p^{(0)}(\mathbf{q},\omega_1,\omega_2)$. This means emission of the term $\gamma^{(v)}$ in the rhs of Eq. (24) that, in its turn, leads to omission of the second pair of terms in the lhs of Eq. (21), which, as mentioned above, correspond directly to the terms in the energy conservation law (18)–(20) including explicitly the scatterer polarization. Thus, since Eq. (30) from [6] is the basic equation for the approach developed by Nieh *et al.*, we must conclude that the ''diffusive Ward identity' derived in Ref. [6] cannot be true.

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